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Contact Process in a Wedge

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Abstract We prove that the supercritical one-dimensional contact process survives in certain wedge-like space-time regions, and that when it survives it couples with the unrestricted contact process started from its upper invariant measure. As an application we show that a type of weak coexistence is possible in the nearest-neighbor "grass-bushes-trees" successional model introduced in Durrett and Swindle (Stoch. Proc. Appl. 37:19–31, 1991).

Keywords Contact process · Grass-bushes-trees

1 Introduction

The contact process of Harris (introduced in [6]) is a well known model of infection spread by contact. The one-dimensional model is a continuous time Markov process ξ_t on $\{0,1\}^{\mathbb{Z}}$. For $x \in \mathbb{Z}$, $\xi_t(x) = 1$ means the individual at site x is infected at time t while $\xi_t(x) = 0$ means the individual is healthy. Infected individuals recover from their infection after an exponential time with mean 1, independently of everything else. Healthy individuals become infected at a rate proportional to the number of infected neighbors. Alternatively, individuals (1's) die at rate one and give birth onto neighboring empty sites (0's) at rate λ. If we let

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$n_i(x,\xi) = \sum_{y:|y-x|=1} 1\{\xi(y) = i\}$, and $\lambda \geq 0$ the infection parameter, then the transitions at x in state ξ are

$$1 \to 0 \text{ at rate } 1 \quad \text{and} \quad 0 \to 1 \text{ at rate } \lambda n_1(x,\xi). \tag{1}$$

When convenient we will identify $\xi \in \{0,1\}^{\mathbb{Z}}$ with $\{x : \xi(x) = 1\}$, and use the notation $\|\xi\|_i = \sum_x 1\{\xi(x) = i\}$.

Let ξ_t^0 denote the contact process with initial state $\xi_0^0 = \{0\}$. The critical value λ_c is defined by

$$\lambda_c = \inf\{\lambda \geq 0 : P(\xi_t^0 \neq \emptyset \text{ for all } t \geq 0) > 0\}. \tag{2}$$

It is well known that $0 < \lambda_c < \infty$, and that in the supercritical case $\lambda > \lambda_c$ there is a unique stationary distribution ν for ξ_t, called the upper invariant measure, with the property

$$\nu\big(\xi : \|\xi\|_1 = \infty\big) = 1.$$

There are also well-defined "edge speeds." Let ξ_0^- (ξ_0^+) be the initial state given by $\xi_0^- = \mathbb{Z}^-$ ($\xi_0^+ = \mathbb{Z}^+$), and define the edge processes

$$r_t = \max\{x : \xi_t^-(x) = 1\} \quad \text{and} \quad l_t = \min\{x : \xi_t^+(x) = 1\}. \tag{3}$$

There is a strictly increasing function $\alpha : (\lambda_c, \infty) \to (0, \infty)$ such that for $\lambda > \lambda_c$

$$\lim_{t\to\infty} \frac{r_t}{t} = \alpha(\lambda) \quad \text{and} \quad \lim_{t\to\infty} \frac{l_t}{t} = -\alpha(\lambda) \text{ a.s.} \tag{4}$$

All of the above facts are contained in Chap. VI of [7] and Part I of [8]. See also Part I of [8] and the Bibliography there for many references to original papers.

The focus of this paper is a version of the contact process in which infection is restricted to certain space-time regions. In particular, we are interested in whether or not survival of the infection is possible in narrow, wedge-like regions, and, given survival, how does such a process behave. For $\mathcal{W} \subset \mathbb{Z} \times [0,\infty)$ define the space-time inhomogeneous $\mathcal{W}$-restricted contact process $\xi_t^{\mathcal{W}}$ as follows. First, set $\xi_t^{\mathcal{W}}(x) = 0$ for all $(x,t) \notin \mathcal{W}$. Second, for $(x,t) \in \mathcal{W}$, replace (1) with

$$1 \to 0 \text{ at rate } 1 \quad \text{and} \quad 0 \to 1 \text{ at rate } \lambda \sum_{y:|y-x|=1} \xi(y) 1_{\mathcal{W}}(y,t), \tag{5}$$

so that infection spreads only between sites in the wedge. We will give an explicit *graphical construction* of $\xi_t^{\mathcal{W}}$ in Sect. 2. Our notation is slightly different from the standard one for which the superscript designates the initial state. For space-time regions $\mathcal{W}$, the notation $\xi_t^{\mathcal{W}}$ indicates the contact process with infection restricted to $\mathcal{W}$, and we will always explicitly state our choice of initial state $\xi_0^{\mathcal{W}}$.

For $0 < \alpha_l < \alpha_r < \infty$ and nonnegative integers $M \geq 0$ define the "wedges" $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$ by

$$\mathcal{W} = \{(x,t) \in \mathbb{Z} \times [0,\infty) : \alpha_l t \leq x \leq M + \alpha_r t\}. \tag{6}$$

In view of (4), we will impose the conditions

$$\lambda > \lambda_c \quad \text{and} \quad 0 < \alpha_l < \alpha_r < \alpha(\lambda). \tag{7}$$

Our first result is that survival in wedges is possible.

Theorem 1 *Assume* (7) *holds*, $\mathcal{W}=\mathcal{W}(\alpha_l,\alpha_r,M)$, *and* $\xi_0^{\mathcal{W}}=[0,M]\cap\mathbb{Z}$. *Then*

$$\lim_{M\to\infty} P(\xi_t^{\mathcal{W}}\neq\emptyset \text{ for all } t\geq 0)=1. \tag{8}$$

When $\xi_t^{\mathcal{W}}$ survives it looks like the unrestricted contact process in equilibrium. To state this more precisely, let

$$r_t^{\mathcal{W}}=\max\{x:\xi_t^{\mathcal{W}}(x)=1\} \quad \text{and} \quad l_t^{\mathcal{W}}=\min\{x:\xi_t^{\mathcal{W}}(x)=1\}, \tag{9}$$

and let $\hat{\xi}_t$ denote the contact process started in its upper invariant measure ν. (That is, $\hat{\xi}_0$ is random with law ν, and given $\hat{\xi}_0$, $\hat{\xi}_t$ makes transitions according to (1).)

Theorem 2 *Assume* $M\geq 1$ *and* (7) *holds. Let* $\mathcal{W}=\mathcal{W}(\alpha_l,\alpha_r,M)$ *and assume* $\xi_0^{\mathcal{W}}=[0,M]\cap\mathbb{Z}$. *On the event* $\{\xi_t^{\mathcal{W}}\neq\emptyset \text{ for all } t\geq 0\}$,

$$\lim_{t\to\infty}\frac{r_t^{\mathcal{W}}}{t}=\alpha_r \quad \text{and} \quad \lim_{t\to\infty}\frac{l_t^{\mathcal{W}}}{t}=\alpha_l \text{ a.s.} \tag{10}$$

Furthermore, $\xi_t^{\mathcal{W}}$ *and* $\hat{\xi}_t$ *can be coupled so that on the event* $\{\xi_t^{\mathcal{W}}\neq\emptyset \text{ for all } t\geq 0\}$,

$$\xi_t^{\mathcal{W}}(x)=\hat{\xi}_t(x) \quad \text{for all } x\in[l_t^{\mathcal{W}},r_t^{\mathcal{W}}] \quad \text{for all large } t \text{ a.s.} \tag{11}$$

Theorem 2 is analogous to results for the unrestricted contact process, see Theorem VI.2.2 in [7].

Remark 3 It is not difficult to extend the argument used in the proof of Theorem VI.3.33 in [7] to show that for $a<b$, $|\hat{\xi}_t\cap[at,bt]|/t\to(b-a)\rho(\lambda)$ a.s. as $t\to\infty$, where $\rho(\lambda)=\nu(\xi:\xi(0)=1)$. Therefore Theorem 2 implies that if $\alpha_l<a<b<\alpha_r$, then when $\xi_t^{\mathcal{W}}$ survives, $|\xi_t^{\mathcal{W}}\cap[at,bt]|/t\to(b-a)\rho(\lambda)$ a.s.

Theorem 1 can be used to obtain information about the "grass-bushes-trees" model (GBT) of [4]. In this model sites are either empty (0), occupied by a bush (1) or occupied by a tree (2). Both 1's and 2's turn to 0's at rate one. The 2's give birth at rate λ_2 on top of 1's and 0's. The 1's give birth at rate λ_1 on top of 0's only, and hence are at a disadvantage compared to 2's. The state space for the process is $\{0,1,2\}^{\mathbb{Z}}$, and the nearest-neighbor version of the model makes transitions at x in state ζ

$$0\to\begin{cases}1 & \text{at rate } \lambda_1 n_1(x,\zeta)\\ 2 & \text{at rate } \lambda_2 n_2(x,\zeta)\end{cases} \quad 1\to\begin{cases}0 & \text{at rate } 1\\ 2 & \text{at rate } \lambda_2 n_2(x,\zeta)\end{cases} \quad 2\to 0 \text{ at rate } 1. \tag{12}$$

A natural question to ask is whether or not coexistence of 1's and 2's is possible. It was shown in [4] that coexistence is possible for a non-nearest neighbor version of the model and appropriate λ_i, where coexistence meant that ζ_t had a stationary distribution μ such that

$$\mu\Big(\{\zeta:\|\zeta\|_i=\infty \text{ for } i=1,2\}\Big)=1. \tag{13}$$

It was also shown in [4] that there is no stationary distribution satisfying (13) in the nearest-neighbor case for *any* choice of the λ_i. Moreover, if there are infinitely many 2's initially then for each site there is a last time at which a 1 can be present. Nevertheless, it is a consequence of Theorem 1 and the construction used in its proof that a form of weak coexistence is possible, even starting from a single 1 and infinitely many 2's.

Corollary 4 *Let ζ_t be the GBT process with initial state ζ_0, where $\zeta_0(x) = 2$ for $x < 0$, $\zeta_0(0) = 1$ and $\zeta_0(x) = 0$ for $x > 0$. For all $\lambda_c < \lambda_2 < \lambda_1$,*

$$P\left(\lim_{t\to\infty} \|\zeta_t\|_1 = \infty\right) > 0. \tag{14}$$

The 2's spread to the right at rate $\alpha(\lambda_2)$, ignoring the 1's, while the 1's try to spread to the right at the faster rate $\alpha(\lambda_1)$. The 1's will be killed by 2's invading from the left, but Theorem 1 shows that they can survive with positive probability by moving off to the right in the space-time region free of 2's.

Remark 5 (1) Corollary 4 is proved by showing that the set of 1's in ζ_t dominates an appropriate wedge-restricted contact process with positive probability. By working harder one can use this idea and Theorem 2 to obtain more information about the set of 1's in ζ_t, but we will not pursue this here. (2) Our model is a stochastic process in an inhomogeneous environment. There are numerous papers in the physics literature regarding such models, see for instance [2, 10] and [9] and more recently in the mathematical biology literature, see [1]. Closer to this paper is the work on non-oriented percolation in various subsets of $\mathbb{Z}^d$ that has been studied in [5] and in [3], but as far as we are aware our results on oriented percolation are new.

In Sect. 2 we give the standard graphical construction due to Harris, then prove Theorem 1 in Sect. 3, Theorem 2 in Sect. 4, and Corollary 4 in Sect. 5.

2 The Graphical Representation

For $x \in \mathbb{Z}$ let $\{T_n^x : n \geq 1\}$ be the arrival times of a Poisson process with rate 1, and for all pairs of nearest-neighbor sites x, y let $\{B_n^{x,y} : n \geq 1\}$ be the arrival times of a Poisson process with rate λ. The Poisson processes T^x, $B^{x,y}$, $x, y \in \mathbb{Z}$, are all independent. At the times T_n^x we put a δ at site x to indicate a death at x, and at the times $B_n^{x,y}$ we draw an arrow from x to y, indicating that a 1 at x will give birth to a 1 at y. For $0 \leq s < t$ and sites x, y we say that there is an active path up from (x, s) to (y, t) if there is a sequence of times $t_0 = s \leq t_1 < t_2 < \cdots < t_n \leq t_{n+1} = t$ and a sequence of sites $x_0 = x, x_1, \ldots, x_n = y$ such that

1. if $n \geq 1$, then for $i = 1, 2 \ldots, n$, $|x_i - x_{i-1}| = 1$ and there is an arrow from x_{i-1} to x_i at time t_i
2. for $i = 0 \ldots, n$, the time segments $\{x_i\} \times [t_i, t_{i+1}]$ do not contain any δ's

By default there is always an active path up from (y, t) to (y, t). For a space-time region $\mathcal{W} \subset \mathbb{Z} \times [0, \infty)$ we define $\xi_t^{\mathcal{W}}$, the contact process restricted to $\mathcal{W}$, as follows. Given an initial state $\xi_0 \subset \{x : (x, 0) \subset \mathcal{W}\}$, set $\xi_t(y) = 0$ for all $(y, t) \notin \mathcal{W}$. If there is a site x with $\xi_0(x) = 1$ and an active path up from $(x, 0)$ to (y, t) lying entirely in $\mathcal{W}$ set $\xi_t^{\mathcal{W}}(y) = 1$, otherwise set $\xi_t^{\mathcal{W}}(y) = 0$. For $\mathcal{W} = \mathbb{Z} \times [0, \infty)$ we will write ξ_t and refer to it as the unrestricted process.

We may also construct the GBT process ζ_t with the above Poisson processes and the help of some additional independent coin flips. Fix $\lambda_c < \lambda_2 < \lambda_1$, and suppose $\lambda = \lambda_1$ in the construction just given. Independently of everything else, label the arrows determined by the B_n^{xy} with a "1-only" sign with probability $(\lambda_1 - \lambda_2)/\lambda_1$. Call an active path up from (x, s) to (y, t) a 2-path if none of its arrows are 1-only arrows. Given ζ_0, we may now construct

ζ_t as follows. (i) For all $t > 0$ and $x \in \mathbb{Z}$, put $\zeta_t(x) = 2$ if for some site y with $\zeta_0(y) = 2$ there is an active 2-path up from $(y, 0)$ to (x, t). (ii) If we have not already set $\xi_t(x) = 2$, put $\zeta_t(x) = 1$ if for some site y with $\zeta_0(y) = 1$ there is an active path up from $(y, 0)$ to (x, t) with the property that no vertical segments in the path contain a point (z, u) such that $\zeta_u(z) = 2$. (iii) For all other (x, t), put $\zeta_t(x) = 0$. A little thought shows that ζ_t is the GBT process with the rates given in (12). The process of 2's is a contact process with infection parameter λ_2, and in the absence of 2's, the process of 1's is a contact process with infection parameter λ_1.

3 Proof of Theorem 1

The Space-Time Regions $\mathcal{Y}_{jk}$ We will modify somewhat the standard approach of constructing a mapping from appropriate space-time regions of the construction just given to an oriented-percolation model with the property that survival of the percolation process implies survival of the contact process. We will call the regions $\mathcal{Y}_{jk}$, they will be defined using the parallelograms of Sect. VI.3 of [7]. We will make use of the following notation: for $x\mathbb{R}^2$ and $A \subset \mathbb{R}^2$, $x + A = \{x + a : a \in A\}$, the translate of A by x.

Let $\mathcal{L}$ be the lattice $\mathcal{L} = \{(j,k) \in \mathbb{Z}^2 : k \geq 0 \text{ and } j+k \text{ is even}\}$ with norm $\|(j,k)\| = 1/2(|j| + |k|)$. Fix $0 < \beta < \alpha/3$ and $M > 0$ so that $M\beta/2$ and $M\alpha$ are integers. Later we will set $\alpha = \alpha(\lambda)$ and take β small. For $(j,k) \in \mathcal{L}$, L_{jk} and R_{jk} are the "large" space-time parallelograms in $\mathbb{Z} \times [0, \infty)$ given by:

$$L_{jk} = (Mj(\alpha - \beta), Mk) + L_{00}, \qquad R_{jk} = (Mj(\alpha - \beta), Mk) + R_{00}$$

where

$$L_{00} = \{(x,t) \in \mathbb{Z} \times [0, M(1 + \beta/\alpha)] : M\beta/2 \leq x + \alpha t \leq 3M\beta/2\}$$
$$R_{00} = \{(x,t) \in \mathbb{Z} \times [0, M(1 + \beta/\alpha)] : -3M\beta/2 \leq x - \alpha t \leq -M\beta/2\}\,.$$

Observe that $L_{jk}(R_{jk})$ is simply a translation of $L_{00}(R_{00})$ by the vector $(Mj(\alpha - \beta), Mk)$. We will also need the "small" parallelograms

$$L_{jk}^{small} = (Mj(\alpha - \beta), Mk) + L_{00}^{small}, \qquad R_{jk}^{small} = (Mj(\alpha - \beta), Mk) + R_{00}^{small}$$

where

$$L_{00}^{small} = \left\{(x,t) \in \mathbb{Z} \times \left[0, M\frac{3\beta}{2\alpha}\right] : M\beta/2 \leq x + \alpha t \leq 3M\beta/2\right\}$$
$$R_{00}^{small} = \left\{(x,t) \in \mathbb{Z} \times \left[0, M\frac{3\beta}{2\alpha}\right] : -3M\beta/2 \leq x - \alpha t \leq -M\beta/2\right\}.$$

It is important to note that $L_{00}^{small} \subset L_{00}$, $R_{00}^{small} \subset R_{00}$, and

$$R_{jk} \cap L_{jk} = R_{jk} \cap L_{jk}^{small} = R_{jk}^{small} \cap L_{jk}\,,$$

as shown in Fig. 1.

We can now define the new objects $\mathcal{Y}_{jk}$ which will be used to construct our oriented percolation process. As is the case with the parallelograms, the $\mathcal{Y}_{jk}$ will be certain translates

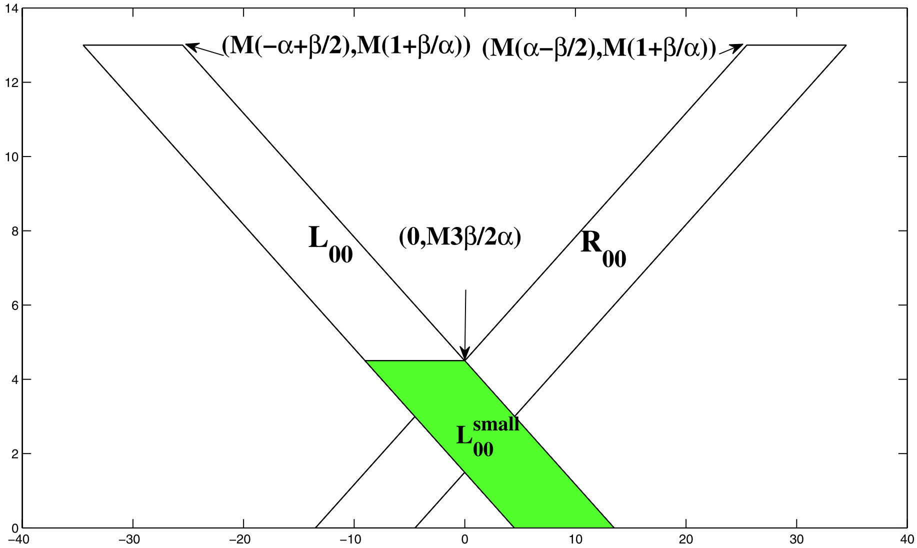

Fig. 1 (Color online) Large parallelograms L_{00} and R_{00}. The *shaded region* is L_{00}^{small}

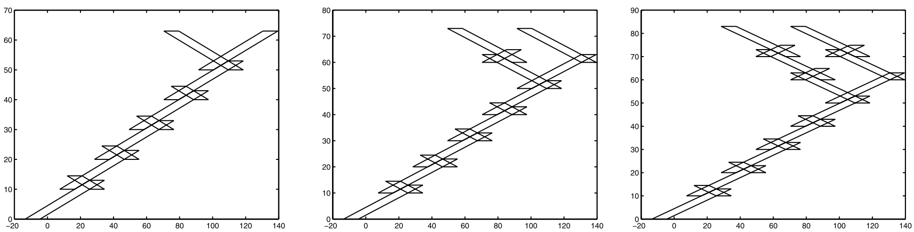

Fig. 2 $\mathcal{Y}_{00}$ with $\ell = 5$, $d = 0, 1, 2$

of $\mathcal{Y}_{00}$, and depend on two fixed integers ℓ, d which satisfy $\ell \geq 2$ and $d \geq 0$ with $\ell > d$ (the dependence on ℓ, d will be surpressed from the notation). We will form $\mathcal{Y}_{00}$ by sticking together ℓ big right parallelograms, connected with appropriate small left parallelograms, and then two branches of d and $d+1$ big left parallelograms connected by small right parallelograms. Figure 2 shows examples of $\mathcal{Y}_{00}$ with parameters $\ell = 5$ and $d = 0, 1, 2$. It seems simplest to define $\mathcal{Y}_{00}$ in stages, beginning with $\mathcal{Y}_{00}^0 = R_{00}$.

1. Attach ℓ big right parallelograms with ℓ small parallelograms to connect them:

$$\mathcal{Y}_{00}^1 = \mathcal{Y}_{00}^0 \cup \left(\bigcup_{i=1}^{\ell} (R_{ii} \cup L_{ii}^{small}) \right).$$

2. Attach one big left parallelogram: $\mathcal{Y}_{00}^2 = \mathcal{Y}_{00}^1 \cup L_{\ell,\ell}$.
3. If $d = 0$ set $\mathcal{Y}_{00} = \mathcal{Y}_{00}^2$. If $d \geq 1$, attach another big left parallelogram:

$$\mathcal{Y}_{00}^3 = \mathcal{Y}_{00}^2 \cup L_{\ell+1,\ell+1}.$$

4. If $d=1$, attach another big left and small right parallelogram:

$$\mathcal{Y}_{00}^4 = \mathcal{Y}_{00}^3 \cup (L_{\ell-1,\ell+1} \cup R_{\ell-1,\ell+1}^{small})$$

and set $\mathcal{Y}_{00} = \mathcal{Y}_{00}^4$. If $d \geq 2$, attach two branches, to reach "height" $\ell + d + 1$, of big left parallelograms with small right parallelograms as connectors:

$$\mathcal{Y}_{00}^4 = \mathcal{Y}_{00}^3 \cup \left(\bigcup_{i=0}^{d-1} (L_{\ell-i,\ell+i} \cup R_{\ell-i,\ell+i}^{small}) \cup (L_{\ell+1-i,\ell+1+i} \cup R_{\ell+1-i,\ell+1+i}^{small}) \right).$$

5. If $d \geq 2$, attach a final big left parallelogram and small right parallelogram:

$$\mathcal{Y}_{00}^5 = \mathcal{Y}_{00}^4 \cup L_{\ell-d,\ell+d} \cup R_{\ell-d,\ell+d}^{small}$$

and put $\mathcal{Y}_{00} = \mathcal{Y}_{00}^5$.

Having defined $\mathcal{Y}_{00}$ we set

$$\mathcal{Y}_{jk} = \big(M[k(\ell-d)+j](\alpha-\beta), Mk(\ell+d+1)\big) + \mathcal{Y}_{00}, \quad (j,k) \in \mathcal{L}.$$

The Percolation Variables U_{jk} Given ℓ, d and the objects $\mathcal{Y}_{jk}$, let $\mathcal{O}_{jk}$ be the event that for every parallelogram $\mathcal{P}$ in $\mathcal{Y}_{jk}$ there is an active path in the graphical representation of the contact process which stays entirely in $\mathcal{P}$ and connects some point in the bottom edge of $\mathcal{P}$ to some point in the top edge of $\mathcal{P}$. Thus on $\mathcal{O}_{jk}$ there is some point in the bottom edge of $\mathcal{Y}_{jk}$ with the property that there are active paths in $\mathcal{Y}_{jk}$ connecting this point to the top edge of every parallelogram in $\mathcal{Y}_{jk}$, and in particular to the top edges of the two top parallelograms $\mathcal{Y}_{jk}$. This means that on $\mathcal{O}_{jk}$ there is a point in the bottom edge of $\mathcal{Y}_{jk}$ and active paths in $\mathcal{Y}_{jk}$ connecting this point to the bottom edges of both $\mathcal{Y}_{j-1,k+1}$ and $\mathcal{Y}_{j+1,k+1}$.

It is a consequence of Lemma VI.3.17 in [7] that $P(\mathcal{O}_{00})$ is close to 1 for large M.

Lemma 6 *For* $0 < \beta < \alpha/3$ *and fixed* ℓ, d, $\lim_{M\to\infty} P(\mathcal{O}_{00}) = 1$.

Proof As in [7] let $\mathcal{E}_{jk}$ be the event that there is an active path in the graphical representation of the contact process which goes from the bottom edge of R_{jk} to the top edge, always staying entirely within R_{jk}, and also that there is an active path from the bottom edge of L_{jk} to the top edge, always staying entirely within L_{jk}. It is clear that the probability of connecting the bottom edge of a small parallelogram to its top edge by an active path staying in the parallelogram is bounded below by $P(\mathcal{E}_{00})$. By Lemma 3.17 in [7], for $0 < \beta < \alpha/3$, $\lim_{M\to\infty} P(\mathcal{E}_{00}) = 1$. In the construction of $\mathcal{Y}_{00}$ there are most $h = 2\ell + 4d$ (if $d \geq 1$) or $h = 2\ell + 1$ (if $d = 0$) parallelograms used. It follows from positive correlations that $P(\mathcal{O}_{00}) \geq P(\mathcal{E}_{jk})^h$, and thus $\lim_{M\to\infty} P(\mathcal{O}_{00}) = 1$ □

For $(j,k) \in \mathcal{L}$ let $U_{jk} = 1_{\mathcal{O}_{jk}}$. Then $P(U_{jk} = 1) = P(\mathcal{O}_{00})$ does not depend on (j,k). Furthermore, the U_{jk} are 1-dependent, meaning that if $I \subset \mathcal{L}$ is such that $\|(j,k)-(j',k')\| > 1$ for all $(j,k) \neq (j',k') \in I$, then the $U_{jk}, (j,k) \in I$ are independent. This is because the corresponding space-time regions $\mathcal{Y}_{jk}, \mathcal{Y}_{j'k'}$ are disjoint. Using the U_{jk} we may construct a 1-dependent oriented percolation process in the usual way. A *path* in $\mathcal{L}$ is a sequence $(j_1,k_1), \ldots, (j_n,k_n)$ of points of $\mathcal{L}$ which satisfies $k_{i+1} = k_i + 1$ and $j_{i+1} = j_i \pm 1$ for all $1 \leq i \leq n-1$. The path is said to be *open* if $U_{j_i,k_i} = 1$ for each $1 \leq i \leq n-1$. It is clear from the properties of the $\mathcal{O}_{jk}$ that if $(j_1,k_1), \ldots, (j_n,k_n)$ is an open path in $\mathcal{L}$ then there must be

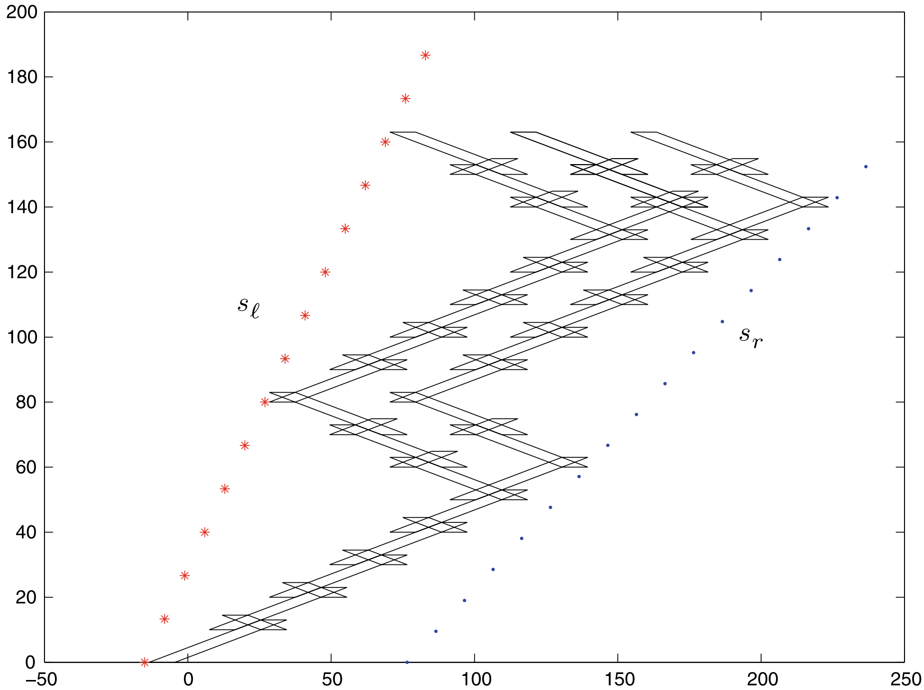

Fig. 3 (Color online) $\mathcal{Y}_{00}, \mathcal{Y}_{1,1}, \mathcal{Y}_{-1,1}$

an active path in the graphical representation from the bottom edge of $\mathcal{Y}_{j_1,k_1}$ to the bottom edge of $\mathcal{Y}_{j_n,k_n}$.

If we let Ω_∞ be the event that there is an infinite open path in $\mathcal{L}$ starting at $(0,0)$, then by Lemma 6 above and Theorem VI.3.19 of [7],

$$\lim_{M\to\infty} P(\Omega_\infty) = 1. \tag{15}$$

Survival of $\xi_t^{\mathcal{W}}$ Let $\mathcal{Y} = \mathcal{Y}(\ell, d, M) = \bigcup_{k=0}^{\infty} \bigcup_{j=-k}^{k} \mathcal{Y}_{jk}$. On Ω_∞ there must be an infinite active path in the graphical representation starting at some $(x,0)$, $x \in [-3M\beta/2, -M\beta/2]$, which lies entirely in $\mathcal{Y}$. Thus if $\mathcal{W}$ is any space-time region such that $\mathcal{Y} \subset \mathcal{W}$, and $\xi_t^{\mathcal{W}}$ is the $\mathcal{W}$-restricted contact process starting from $\{x : (x,0) \subset \mathcal{W}\}$, then $\xi_t^{\mathcal{W}} \neq \emptyset\ \forall t \geq 0$ on Ω_∞. We will prove the following.

Claim Assume (7) holds and $\alpha = \alpha(\lambda)$. Then there exists $0 < \beta < \alpha/3$ and integers ℓ', d' such that for all $M > 0$,

$$\mathcal{Y}(\ell', d', M/\alpha(\ell'+3)) \subset \mathcal{W}(\alpha_l, \alpha_r, M) - (M/(\ell'+3), 0). \tag{16}$$

Given (16), it follows from translation invariance and (15) that

$$P(\xi_t^{\mathcal{W}(\alpha_l,\alpha_r,M)} \neq \emptyset\ \forall t \geq 0) \geq P(\Omega_\infty) \to 1 \quad \text{as } M \to \infty,$$

proving (8).

To prove (16) we first suppose that ℓ, d, are positive integers with $d < \ell$ and $M > 0$. For $(j,k) \in \mathcal{L}$, the left upper corner of L_{jk} is $(M(j(\alpha-\beta)-\alpha-\beta/2), M(k+1+\beta/\alpha))$, and the right bottom corner of L_{jk} is $(M(j(\alpha-\beta)+3\beta/2), Mk)$. A little thought shows that $\mathcal{Y}$ must be contained in the space-time region bounded by the following two lines and the x-axis. The first line connects the leftmost point of the top edge of $\mathcal{Y}_{00}$ with the leftmost point of the top edge of $\mathcal{Y}_{-1,1}$, which are the left upper corner of $L_{\ell-d,\ell+d}$ and the left upper corner of $L_{2(\ell+d)-1,2(\ell+d)+1}$, namely, the points

$$(M((\ell-d)(\alpha-\beta)-\alpha-\beta/2), M(\ell+d+1+\beta/\alpha))$$

and

$$(M(2(\ell-d)(\alpha-\beta)-2\alpha+\beta/2), M(2(\ell+d+1)+\beta/\alpha)).$$

The slope of this line is

$$s_l = \frac{\ell+d+1}{\ell-d-1}\frac{1}{\alpha-\beta} \tag{17}$$

and it contains the point $(x_l, 0)$ where $x_l = -M(3\beta/2 + \beta/\alpha s_l)$. The second line connects the rightmost point of $\mathcal{Y}_{00}$ with the rightmost point of $\mathcal{Y}_{1,1}$, the bottom right corner of $L_{\ell+1,\ell+1}$ and the bottom right $L_{2(\ell+1)-d,2(\ell+1)+d}$, namely, the points

$$(M((\ell+1)(\alpha-\beta)+3\beta/2), M(\ell+1))$$

and

$$(M((2(\ell+1)-d)(\alpha-\beta)+3\beta/2), M(2(\ell+1)+d)).$$

The slope of this line is

$$s_r = \frac{\ell+d+1}{\ell-d+1}\frac{1}{\alpha-\beta} \tag{18}$$

and it contains the point $(x_r, 0)$ where $x_r = M((\ell+1)(\alpha-\beta-1/s_r)+3\beta/2)$.

This analysis shows that $\mathcal{Y}(\ell,d,M)$ is contained in the wedge $\mathcal{W}(1/s_l, 1/s_r, M') + (x_l, 0)$, where $M' = x_r - x_l$. A little algebra shows that $-M\alpha < x_l < x_r < M\alpha(\ell+2)$, and thus

$$\mathcal{Y}(\ell,d,M) \subset \mathcal{W}(1/s_l, 1/s_r, M\alpha(\ell+3)) - (M\alpha, 0). \tag{19}$$

We now set $s_\ell = 1/\alpha_\ell$, $s_r = 1/\alpha_r$ and solve (17) and (18) for d and ℓ, obtaining

$$\ell = \frac{s_r(s_l(\alpha-\beta)+1)}{s_l - s_r}, \qquad d = \frac{s_l(s_r(\alpha-\beta)-1)}{s_l-s_r}. \tag{20}$$

Unfortunately, ℓ, d need not be integers. To deal with this problem we first note that if $s_l \geq s_l' > s_r$ then for any M, the wedge $\mathcal{W}(\alpha_l, \alpha_r, M)$ contains the narrower wedge $\mathcal{W}(1/s_\ell', 1/s_r, M)$. If we can find s_ℓ' and $0 < \beta < \alpha/3$ such that

$$\ell' = \frac{s_r(s_l'(\alpha-\beta)+1)}{s_l'-s_r} \quad \text{and} \quad d' = \frac{s_l'(s_r(\alpha-\beta)-1)}{s_l'-s_r} \tag{21}$$

are both integers, then (16) follows from (19).

We can find s_ℓ', β as follows. Let $m_0 = 3/\alpha s_r$ and take any integer $m > m_0$ such that $s_r\frac{m}{m-1} < s_l$. Put $s_l' = s_r\frac{m}{m-1}$, so that $s_l > s_l' > s_r$. Since $m > 3/\alpha s_r$, $1/3\alpha m s_r > 1$ and the

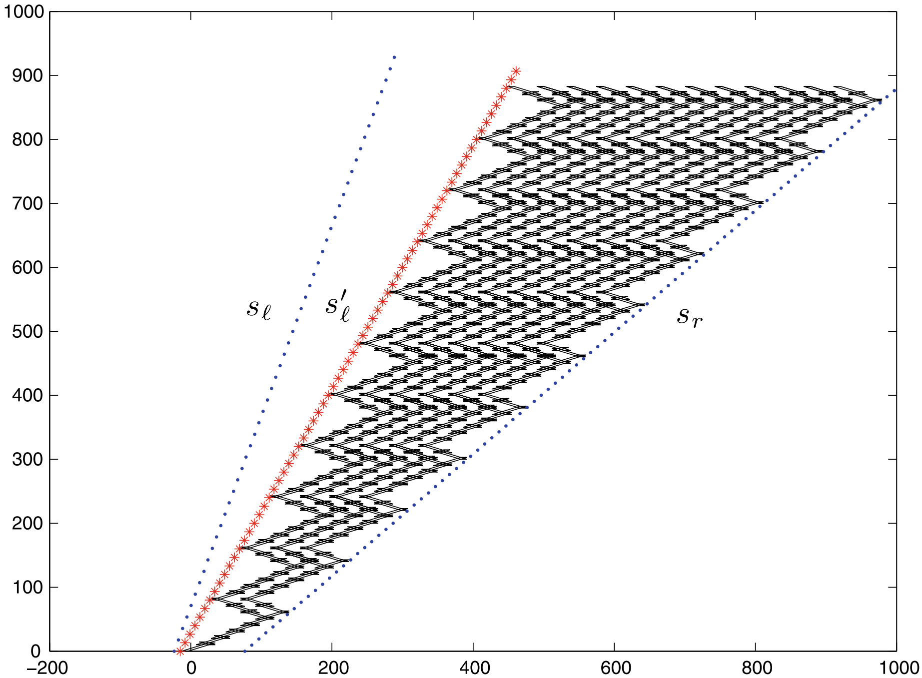

Fig. 4 (Color online) Wedge containing $\mathcal{Y}$

interval $(\frac{2}{3}\alpha m s_r, \alpha m s_r)$ must contain at least one integer. Since $\alpha s_r > 1$, the right endpoint of this interval is greater than m. Choose any integer $c \geq m$ from the interval and put $\beta = \alpha - \frac{c}{m s_r}$. Then $0 < \beta < \alpha/3$ and $s_r(\alpha - \beta) = c/m$. A little algebra shows that ℓ', d' given in (21) are the integers $\ell' = c + m - 1$, $d' = c - m$, and we are done.

4 Proof of Theorem 2

We begin by analyzing the rightmost particle. Let $\mathcal{W}(\alpha_r, M) = \{(x,t) : t \geq 0, x \in (-\infty, M + \alpha_r t] \cap \mathbb{Z}\}$ and consider the restricted contact process $\xi_t^{\mathcal{W}(\alpha_r, M)}$ with initial state $\xi_0^{\mathcal{W}(\alpha_r, M)} = (-\infty, M] \cap \mathbb{Z}$. Let $\bar{r}_t^M$ be the corresponding right-edge process, $\bar{r}_t^M = \max\{x : \xi_t^{\mathcal{W}(\alpha_r, M)}(x) = 1\}$. We claim that for every M,

$$\lim_{t\to\infty} \frac{\bar{r}_t^M}{t} = \alpha_r \quad \text{a.s.} \tag{22}$$

By stationarity of the basic Poisson processes, the law of $\bar{r}^{M+1}$ is the same as the law of $\bar{r}^M + 1$, so it suffices to prove (22) for $M = 0$. By construction, $\limsup_{t\to\infty} \bar{r}_t^0/t \leq \alpha_r$ a.s.

For the lower bound, fix $0 < \varepsilon < \alpha_r$ and consider the region $\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M)$ and the restricted contact process $\xi_t^{\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M)}$ with initial state $[0, M] \cap \mathbb{Z}$. For fixed $\delta > 0$, Theorem 1 implies there exists M_0 such that the event $\{\xi_t^{\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M_0)} \neq \emptyset\ \forall t \geq 0\}$ has probability at least $1 - \delta$. On this event, since $\xi_t^{\mathcal{W}(\alpha_r - \varepsilon, \alpha_r, M_0)} \subset \xi_t^{\mathcal{W}(\alpha_r, M_0)}$, $\liminf_{t\to\infty} \bar{r}_t^{M_0}/t \geq \alpha_r - \varepsilon$. Consequently,

$$P(\liminf_{t\to\infty} \bar{r}_t^0/t \geq \alpha_r - \varepsilon) = P(\liminf_{t\to\infty} \bar{r}_t^{M_0}/t \geq \alpha_r - \varepsilon) \geq 1 - \delta.$$

This shows $P(\liminf_{t\to\infty} \bar{r}_t^0/t \geq \alpha_r - \varepsilon) = 1$ for every $\varepsilon > 0$, and together with the previous limsup bound shows $\bar{r}_t^0/t \to \alpha_r$ a.s., establishing (22).

It is a consequence of the nearest-neighbor interaction mechanism that for any $\alpha_l < \alpha_r$ and M, with $\mathcal{W} = \mathcal{W}(\alpha_l, \alpha_r, M)$,

$$\xi_t^{\mathcal{W}}(x) = \xi_t^{\mathcal{W}(\alpha_r, M)}(x) \quad \forall x \in [l_t^{\mathcal{W}}, r_t^{\mathcal{W}}] \quad \text{on} \quad \{\xi_t^{\mathcal{W}} \neq \emptyset\}.$$

This implies $r_t^{\mathcal{W}} = \bar{r}_t^M$ on $\{\xi_t^{\mathcal{W}} \neq \emptyset\}$, and so by (22), $\lim_{t\to\infty} r_t^{\mathcal{W}}/t = \alpha_r$ on the event $\{\xi_t^{\mathcal{W}} \neq \emptyset \ \forall t \geq 0\}$. We omit the similar argument for $\lim_{t\to\infty} l_t^{\mathcal{W}}/t = \alpha_l$.

For (11), let $\xi_t^{\mathbb{Z}}$ denote the unrestricted process with initial state $\xi_0^{\mathbb{Z}} = \mathbb{Z}$, and let $\hat{\xi}_t$ be the unrestricted process constructed in Sect. 2 started in the invariant measure ν. (That is, $\hat{\xi}_0$ is random with law ν, independent of the Poisson processes, and given $\hat{\xi}_0$ the construction of Sect. 2 is used.) We observe again that the nearest-neighbor interaction implies

$$\xi_t^{\mathbb{Z}}(x) = \xi_t^{\mathcal{W}}(x) \quad \forall x \in [l_t^{\mathcal{W}}, r_t^{\mathcal{W}}] \quad \text{on} \quad \{\xi_t^{\mathcal{W}} \neq \emptyset \ \forall t \geq 0\}.$$

Standard exponential estimates for $P(\xi_t^{\mathbb{Z}}(x) \neq \hat{\xi}_t(x)) = P(\xi_t^{\mathbb{Z}}(x) = 1) - P(\hat{\xi}_t(x) = 1)$, a "filling in" argument and Borel-Cantelli (see Theorem I.2.30 of [8]) imply that for any $A > 0$,

$$P(\xi_t^{\mathbb{Z}} = \hat{\xi}_t \text{ on } [-At, At] \text{ for all large } t) = 1$$

Combining the above with (10) gives (11).

5 Proof of Corollary 4

We will make use of the graphical construction in Sect. 2 and define independent events $\Omega_1, \Omega_2, \Omega_3$, each with positive probability, and such that $\|\zeta_t\|_1 \to \infty$ as $t \to \infty$ on their intersection.

First, since $\alpha(\lambda)$ is strictly increasing we may choose $\alpha(\lambda_2) < \alpha_l < \alpha_r < \alpha(\lambda_1)$. Fix $M > 2$ and write $\mathcal{W}$ for $\mathcal{W}(\alpha_l, \alpha_r, M)$. The first event is

$$\Omega_1 = \{\text{there is no active 2-path from any } (x, 0),\ x < 0, \text{ to any point of } \mathcal{W}(\alpha_l, \alpha_r, M)\}.$$

Since the process of 2's is a contact process with parameter λ_2, and $\alpha(\lambda_2) < \alpha_l$, it follows from (4) that Ω_1 has positive probability.

For the second event, choose $x_0 \in \mathbb{Z}$ and $t_0 > 0$ such that $x_0 = \alpha_l t_0$ and $(x, t_0) \subset \mathcal{W}$ for all $x \in [x_0, x_0 + M] \cap \mathbb{Z}$. Since $M > 2$ the event,

$$\Omega_2 = \{\text{there is an active path in } \mathcal{W} \text{ from } (0,0) \text{ to each of } (x, t_0),\ x \in [x_0, x_0 + M] \cap \mathbb{Z}\}$$

has positive probability.

For the third event, define, for $t \geq t_0$,

$$\begin{aligned} A_t = \{y : &\text{ there is an infinite active path in } \mathcal{W} \text{ from } (x, t_0) \text{ to } (y, t) \\ &\text{for some } x \in [x_0, x_0 + M] \cap \mathbb{Z}\ \} \end{aligned}$$

and put $\Omega_3 = \{|A_t| \to \infty \text{ as } t \to \infty\}$. It follows from Theorems 1 and 2 that Ω_3 has positive probability.

The events Ω_i are independent since they are defined in terms of our Poisson processes over disjoint space-time regions. Furthermore, it is easy to see from Remark 3 that $\|\zeta_t\|_1 \to \infty$ on their intersection, so we are done.

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